

ON CERTAIN EQUIDIMENSIONAL POLYMATROIDAL IDEALS

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ABSTRACT. The class of equidimensional polymatroidal ideals is studied. In particular, we show that an unmixed polymatroidal ideal is connected in codimension one if and only if it is Cohen-Macaulay. Especially a matroidal ideal is connected in codimension one precisely when it is a squarefree Veronese ideal. As a consequence we indicate that for polymatroidal ideals, Serre's condition (S_n) for some $n \geq 2$ is equivalent to Cohen-Macaulay property. We also give a classification of generalized Cohen-Macaulay polymatroidal ideals.

1. INTRODUCTION

Throughout this paper we consider monomial ideals of the polynomial ring $S = k[x_1, \dots, x_n]$ over a field k , and $\mathfrak{m} = (x_1, \dots, x_n)$ denotes the unique homogeneous maximal ideal. The Cohen-Macaulay polymatroidal ideals are classified by Herzog and Hibi [6], into the principal ideals, the Veronese ideals, and the squarefree Veronese ideals. As mentioned in [6], it is natural and interesting to classify all unmixed polymatroidal ideals. Recall that an ideal I is called unmixed if all prime ideals in $\text{Ass}(S/I)$ have the same height. If all minimal prime ideals of I have the same height, then I is called equidimensional. Obviously an unmixed ideal I is equidimensional and the converse holds precisely when $\text{Min}(S/I) = \text{Ass}(S/I)$. In particular a squarefree monomial ideal is equidimensional if and only if it is unmixed.

In this paper we study certain classes of equidimensional polymatroidal ideals. After giving some preliminary concepts and results in Section 2, we study the polymatroidal ideals connected in codimension one, in Section 3. Consider the Zarisky topology on $\text{Spec}(S/I)$ for a monomial ideal I . $\text{Spec}(S/I)$ is a connected space with this topology. The ideal I is called connected in codimension one, if $\text{Spec}(S/I)$ remains connected after removing closed subsets with codimension bigger than one [5]. This property can be expressed in terms of minimal prime ideals of I , and implies that I is equidimensional (see Remark 3.2). From combinatorial point of view, a squarefree monomial ideal is connected in codimension one, if it is the Stanley-Reisner ideal of a strongly connected simplicial complex.

As mentioned in Remark 3.4, Cohen-Macaulay ideals are connected in codimension one. The aim of this section is to find when the converse holds true for polymatroidal ideals. Theorem 3.6 states that matroidal ideals connected in codimension one, are precisely squarefree Veronese ideals and thus Cohen-Macaulay. We extend this result to unmixed polymatroidal ideals in Theorem 3.9, the essential result of this section.

2010 *Mathematics Subject Classification.* 13C05, 13C14, 05B35, 05E40.

Key words and phrases. Connected in codimension one, equidimensional ideals, generalized Cohen-Macaulay, polymatroidal ideals, unmixed ideals.

Somayeh Bandari was in part supported by a grant from IPM (No. 92130020)

Raheleh Jafari was in part supported by a grant from IPM (No. 92130420).

The main consequence of this result is Corollary 3.11 which asserts that for polymatroidal ideals satisfying Serre's condition (S_n) for some $n \geq 2$ is equivalent to being Cohen-Macaulay.

The unmixed polymatroidal ideals have also been studied by Vlădoiu in [13]. He shows that an ideal of Veronese type is unmixed if and only if it is Cohen-Macaulay. Our second target is to find equidimensional polymatroidal ideals which are not Cohen-Macaulay, in Section 4. We show that a polymatroidal ideal generated in degree 2, is equidimensional if and only if it is generalized Cohen-Macaulay (see Proposition 4.2) and Example 4.9(iii) is a non-Cohen-Macaulay ideal in this class. An unmixed polymatroidal ideal generated in degree $d > 2$, is not necessarily generalized Cohen-Macaulay (see Example 4.3). In the case of matroidal ideals, Theorem 4.5 states that generalized Cohen-Macaulay matroidal ideals generated in degree $d > 2$, are precisely Cohen-Macaulay matroidal ideals.

By [11, Proposition 5], the polymatroidal ideal I generated in degree d has an irredundant primary decomposition either of the form $I = J \cap \mathfrak{m}^0$ or $I = J \cap \mathfrak{m}^d$. The classification of generalized Cohen-Macaulay polymatroidal ideals, stated in Theorem 4.8, indicates that a fully supported monomial ideal $I = J \cap \mathfrak{m}^s$ generated in degree d with $s \in \{0, d\}$, is a generalized Cohen-Macaulay polymatroidal ideal if and only if one of the following statements holds true:

- a) J is a Cohen-Macaulay polymatroidal ideal i.e. J is either a principal ideal, a Veronese ideal, or a squarefree Veronese ideal.
- b) $J = \mathfrak{p}_1^{a_1} \cap \cdots \cap \mathfrak{p}_r^{a_r}$ is equidimensional and $\mathfrak{p}_i + \mathfrak{p}_j = \mathfrak{m}$ for all $i \neq j$.
- c) J is an unmixed matroidal ideal of degree 2.

There are examples illustrating the significance of each of the items in the above characterization and showing that none of them can be removed, see Examples 4.9 and 4.10.

2. PRELIMINARIES

Throughout this paper $S = k[x_1, \dots, x_n]$ is the polynomial ring over a field k with the unique homogenous maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. For a monomial ideal I of S , the minimal set of monomial generators of I is denoted by $G(I)$ and $\text{supp}(I) := \{x_i; 1 \leq i \leq n, x_i | u \text{ for some } u \in G(I)\}$. We call the monomial ideal I *fully supported* if $\text{supp}(I) = \{x_1, \dots, x_n\}$. An ideal I is said to be *unmixed* if all associated prime ideals of I have the same height and is called *equidimensional* if all minimal prime ideals have the same height.

A monomial ideal I is called a *polymatroidal* ideal, if it is generated in a single degree with the exchange property that for any two elements $u, v \in G(I)$ with $\deg_{x_i}(u) > \deg_{x_i}(v)$, there exists an index j with $\deg_{x_j}(u) < \deg_{x_j}(v)$ such that $x_j(u/x_i) \in G(I)$. It is easy to see that a monomial ideal I is polymatroidal if and only if for all monomials $u, v \in G(I)$ with $\deg_{x_i}(u) > \deg_{x_i}(v)$ for some i , there exists an integer j such that $\deg_{x_j}(v) > \deg_{x_j}(u)$ and $x_j(u/x_i) \in I$. A squarefree polymatroidal ideal is called a *matroidal* ideal.

Recall that any polymatroidal ideal I has a linear resolution by [9, Lemma 1.3] and [3, Lemma 4.1]. As a consequence the Castelnuovo-Mumford regularity of I is equal to d , where I is generated in degree d and we have the following presentation for I which we will use it frequently in our approach.

Proposition 2.1. [11, Proposition 5] *For a polymatroidal ideal $I \subset S$ with $\text{Ass}(S/I) \setminus \{\mathfrak{m}\} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$, there are integers $a_i > 0$ and $s \geq 0$ such that $I = \mathfrak{p}_1^{a_1} \cap \dots \cap \mathfrak{p}_r^{a_r} \cap \mathfrak{m}^s$ and I is generated in degree s , when $s > 0$.*

The following observation shows that an unmixed polymatroidal ideal generated in degree 2, is not very far from a matroidal ideal.

Lemma 2.2. *Let I be a fully supported polymatroidal ideal of S , generated in degree 2. If I is unmixed, then I is a matroidal ideal or $I = \mathfrak{m}^2$.*

Proof. If $|\text{Ass}(S/I)| = 1$, then the result is clear. Otherwise, let $I = \mathfrak{p}_1^{a_1} \cap \dots \cap \mathfrak{p}_r^{a_r}$ be the minimal primary decomposition mentioned in Proposition 2.1. Since $\text{ht}(\mathfrak{p}_i) = \text{ht}(\mathfrak{p}_j)$ for all $i \neq j$, there exist $x_i \in \mathfrak{p}_i \setminus \mathfrak{p}_j$ and $x_j \in \mathfrak{p}_j \setminus \mathfrak{p}_i$. Therefore $x_i^{a'_i} x_j^{a'_j} | u$ for $a'_i \geq a_i \geq 1$, $a'_j \geq a_j \geq 1$ and some $u \in G(I)$. Now, since $\deg(u) = 2$, we have that $a_i = a_j = 1$ for all $i \neq j$ and so I is a matroidal ideal. \square

The unmixed condition is necessary in the above lemma. For instance consider the equidimensional polymatroidal ideal $I = (x_1^2, x_1x_2, x_1x_3, x_2x_3) = (x_1, x_2) \cap (x_1, x_3) \cap (x_1, x_2, x_3)^2$. The point in this example is that I contains a pure power of a variable x_1 but not any other powers x_2^2 or x_3^2 . The following result shows that it can not happen if the ideal is unmixed.

Proposition 2.3. *Let I be an unmixed fully supported polymatroidal ideal of S , generated in degree d . If $x_j^d \in I$ for some $1 \leq j \leq n$, then $I = \mathfrak{m}^d$.*

Proof. Let $I = \mathfrak{p}_1^{a_1} \cap \dots \cap \mathfrak{p}_r^{a_r}$ be the minimal primary decomposition. Since $x_j^d \in I$ for some j , we have $x_j^d = x_j^{\max\{a_i; i=1, \dots, r\}}$. For simplicity, assume that $d = a_1$. Since $I = \mathfrak{p}_1^d \cap \dots \cap \mathfrak{p}_r^{a_r}$ is generated in degree d and $\mathfrak{p}_i \not\subseteq \mathfrak{p}_1$ for all $i = 2, \dots, r$, we get $I = \mathfrak{p}_1^d = \mathfrak{m}^d$. \square

The unmixed polymatroidal ideals which appear in the above statement, powers of the maximal ideal \mathfrak{m} , are called Veronese ideals. In other words the (squarefree) Veronese ideal of degree d in the variables x_{i_1}, \dots, x_{i_r} is the ideal of S which is generated by all (squarefree) monomials in x_{i_1}, \dots, x_{i_r} of degree d .

Theorem 2.4. [6, Theorem 4.2] *A polymatroidal ideal I is Cohen-Macaulay if and only if I is a principal ideal, a Veronese ideal or a squarefree Veronese ideal.*

As a generalization of Veronese ideals, if the ideal I is generated by all monomials u of degree d such that $\deg_{x_i}(u) \leq a_i$ for some integers $a_i \geq 0$, the ideal I is denoted by $I_{d; a_1, \dots, a_n}$ and is called an ideal of Veronese type. Ideals of Veronese type are obviously polymatroidal. If I is an ideal of Veronese type, then $\text{Min}(S/I) = \text{Ass}(S/I)$ if and only if I is unmixed if and only if I is Cohen-Macaulay, see [13, Theorem 3.4].

Let \mathfrak{p} be a monomial prime ideal of S . Then $\mathfrak{p} = \mathfrak{p}_{\{i_1, \dots, i_t\}}$ where $\{i_1, \dots, i_t\} = [n] \setminus \{i; x_i \in \text{supp}(\mathfrak{p})\}$ and $IS_{\mathfrak{p}} = JS_{\mathfrak{p}}$, where J is the monomial ideal obtained from I by the substitution $x_i \mapsto 1$ for all $i = i_1, \dots, i_t$. The ideal J is called the monomial localization of I with respect to \mathfrak{p} and is denoted by $I(\mathfrak{p})$. The following easy observation is a crucial point in using monomial localization as an effective tool.

Remark 2.5. *Let $I = \cap_{i=1}^r Q_i$ be a primary decomposition of a monomial ideal I .*

- a) $I(\mathfrak{p}_{\{1, \dots, t\}}) = \cap_{i \in T} Q_i$ where $T = \{i; 1 \leq i \leq r, \text{supp}(Q_i) \cap \{x_1, \dots, x_t\} = \emptyset\}$.

- b) If I is generated in single degree d and $I(\mathfrak{p}_{\{i\}})$ is generated in single degree d_i , then $d_i = d - a_i$ where $a_i = \max\{\deg_{x_i}(u); u \in G(I)\}$ and

$$G(I(\mathfrak{p}_{\{i\}})) = \left\{ \frac{u}{x_i^{a_i}}; u \in G(I) \text{ and } x_i^{a_i} | u \right\}.$$

3. POLYMATROIDAL IDEALS CONNECTED IN CODIMENSION ONE

In this section we study the Cohen-Macaulay property of polymatroidal ideals from topological point of view. Let I be a monomial ideal of S and consider the Zarisky topology on $\text{Spec}(S/I)$. Recall that the closed subsets in this topology are the sets $V(J) = \{\mathfrak{q}; \mathfrak{q} \in \text{Spec}(S) \text{ and } J \subseteq \mathfrak{q}\}$, where $J \supseteq I$ is an ideal of S . The irreducible components of $\text{Spec}(S/I)$ are the closed sets $V(\mathfrak{p})$, where \mathfrak{p} is a minimal prime ideal of I . $\text{Spec}(S/I)$ with this topology is a connected space. The ideal I is called *connected in codimension one*, if $\text{Spec}(S/I)$ remains connected after removing closed subsets with codimension bigger than one [5]. Since the codimension of $V(\mathfrak{p})$ is equal to $\text{ht}(\mathfrak{p}) - \text{ht}(I)$ for all prime ideals $\mathfrak{p} \supseteq I$, we have the following definition by [5, Proposition 1.1].

Definition 3.1. A monomial ideal $I \subset S$ with height h , is connected in codimension one, if for any pair of distinct prime ideals $\mathfrak{p}, \mathfrak{q} \in \text{Min}(S/I)$ there exists a sequence of minimal prime ideals $\mathfrak{p} = \mathfrak{p}_1, \dots, \mathfrak{p}_r = \mathfrak{q}$ such that $|G(\mathfrak{p}_i + \mathfrak{p}_{i+1})| = h + 1$, for all $1 \leq i \leq r - 1$.

Remark 3.2. By the above definition it is clear that a monomial ideal connected in codimension one is equidimensional and so $|G(\mathfrak{p}_i) \cap G(\mathfrak{p}_{i+1})| = h - 1$, for all $1 \leq i \leq r - 1$. Since for a squarefree monomial ideal I , all associated prime ideals are minimal, being equidimensional is equivalent to being unmixed. Thus if a squarefree monomial ideal I is connected in codimension one, then I is unmixed.

Remark 3.3. In the context of Hartshorne [5], an ideal $I \subset S$ is called locally connected in codimension one if all localizations $I_{\mathfrak{p}}$ are connected in codimension one where $\mathfrak{p} \in V(I)$. If I is a monomial ideal and $I_{\mathfrak{m}}$ is connected in codimension one, then I is connected in codimension one.

From combinatorial point of view, a pure simplicial complex Δ is said to be strongly connected or connected in codimension one, if for any two facets F and G , there is a sequence of facets $F = F_1, F_2, \dots, F_r = G$ such that $\dim(F_i \cap F_{i+1}) = \dim \Delta - 1$ or equivalently $\dim(F_i \cup F_{i+1}) = \dim \Delta + 1$, for each $1 \leq i \leq r - 1$. A squarefree monomial ideal is connected in codimension one, if it is the Stanley-Reisner ideal of a strongly connected simplicial complex.

Remark 3.4. Let I be a Cohen-Macaulay monomial ideal. Then I is connected in codimension one, by [5, Corollary 2.4] and Remark 3.3. Another way to see this fact is observing that \sqrt{I} is also Cohen-Macaulay by [10, Theorem 2.6]. So according to [7, Lemma 9.1.12], I is connected in codimension one, since $\text{Min}(S/I) = \text{Min}(S/\sqrt{I})$.

Obviously an unmixed principal ideal is connected in codimension one. As an easy way to construct a monomial ideal connected in codimension one, we may consider I as the intersection of all prime ideals generated by $h = \text{ht}(I)$ variables. It is indeed the squarefree Veronese ideal generated in degree $d = n - h + 1$ [2, Theorem 3.4]. From

another point of view, I is Cohen-Macaulay by Theorem 2.4 and hence I is connected in codimension one by the above remark.

In Theorem 3.6, we show that all matroidal ideals connected in codimension one, are precisely the squarefree Veronese ideals. As a key point of our proof, we need the following simple characterization which in the case $t = 2$ is also proved by a different method in [2, Lemma 2.3].

Lemma 3.5. *Let I be a matroidal ideal and $T = \{x_1, \dots, x_t\} \subseteq \text{supp}(I)$. If for any $t - 1$ elements $x_{j_1}, \dots, x_{j_{t-1}}$ of T , $x_{j_1} \cdots x_{j_{t-1}} | u$ for some $u \in G(I)$, then the following statements are equivalent.*

- a) $x_1 \cdots x_t \nmid u$ for all $u \in G(I)$.
- b) $I(\mathfrak{p}_{\{1, \dots, t\}}) = I(\mathfrak{p}_{\{j_1, \dots, j_{t-1}\}})$ for all $\{x_{j_1}, \dots, x_{j_{t-1}}\} \subseteq T$.
- c) $|\text{supp}(\mathfrak{p}) \cap \{x_1, \dots, x_t\}| \neq 1$ for all $\mathfrak{p} \in \text{Ass}(S/I)$.

Proof. (a) \Rightarrow (b): By [8, Corollary 3.2] any monomial localization of I is again matroidal and so it is generated in a single degree. Since $x_{j_1} \cdots x_{j_{t-1}} | u$ for some $u \in G(I)$, we have $I_j = I(\mathfrak{p}_{\{j_1, \dots, j_{t-1}\}})$ is generated in degree $d - t + 1$ where d is the degree of the generators of I . Indeed,

$$G(I_j) = \left\{ \frac{u}{x_{j_1} \cdots x_{j_{t-1}}}; u \in G(I) \text{ and } x_{j_1} \cdots x_{j_{t-1}} | u \right\}.$$

On the other hand $x_1 \cdots x_t \nmid u$ for all $u \in G(I)$, therefore $x \notin \text{supp}(I_j)$ for $x \in T \setminus \{x_{j_1}, \dots, x_{j_{t-1}}\}$, and it follows (b).

(b) \Rightarrow (c): Assume that $x_i \in \mathfrak{p}$ for some $\mathfrak{p} \in \text{Ass}(S/I)$ and $1 \leq i \leq t$. Therefore $\mathfrak{p} \notin \text{Ass}(S/I(\mathfrak{p}_{\{1, \dots, t\}})) = \text{Ass}(S/I(\mathfrak{p}_{\{1, \dots, i-1, i+1, \dots, t\}}))$, that is $x_j \in \mathfrak{p}$ for some $1 \leq j \neq i \leq t$.

(c) \Rightarrow (a): Assume that $x_1 \cdots x_t | u$ for some $u \in G(I)$. Then $x_t \in \text{supp}(I(\mathfrak{p}_{\{1, \dots, t-1\}}))$ and so there exists a prime ideal $\mathfrak{p} \in \text{Ass}(S/I(\mathfrak{p}_{\{1, \dots, t-1\}}))$ such that $x_t \in \mathfrak{p}$. Now (c) implies that $x_i \in \mathfrak{p}$ for some $1 \leq i \leq t - 1$ which is a contradiction. \square

Now we are able to classify all matroidal ideals connected in codimension one.

Theorem 3.6. *Let I be a monomial ideal. Then I is a matroidal ideal connected in codimension one if and only if I is a squarefree Veronese ideal.*

Proof. If I is squarefree Veronese ideal, then I is matroidal ideal and connected in codimension one by the explanation after Remark 3.4. Assume that I is a matroidal ideal generated in degree d and is connected in codimension one. We use induction on i , $1 \leq i \leq d$ to show that for any set $\{x_{j_1}, \dots, x_{j_i}\} \subseteq \text{supp}(I)$, there exists $u \in G(I)$ such that $x_{j_1} \cdots x_{j_i} | u$.

Our claim is trivial for $i = 1$. Assume that it's true for $i = t - 1$ and assume contrary that $t \leq d$ and $\{x_1, \dots, x_t\} \subseteq \text{supp}(I)$ and $x_1 \cdots x_t \nmid u$ for all $u \in G(I)$. By induction assumption, for any subset $\{x_{r_1}, \dots, x_{r_{t-1}}\}$ of $t - 1$ elements of $\{x_1, \dots, x_t\}$, $x_{r_1} \cdots x_{r_{t-1}} | u$ for some $u \in G(I)$. Note that by Lemma 3.5, $I(\mathfrak{p}_{\{1, \dots, t\}}) = I(\mathfrak{p}_{\{1, \dots, t-1\}})$ and $I(\mathfrak{p}_{\{1, \dots, t-1\}}) \neq S$, since $t - 1 < d$ and $x_1 \cdots x_{t-1} | u$ for some $u \in G(I)$. Hence there exists $\mathfrak{q} \in \text{Ass}(S/I)$ such that $\{x_1, \dots, x_t\} \cap \text{supp}(\mathfrak{q}) = \emptyset$. Let $\mathfrak{p} \in \text{Ass}(S/I)$ with $x_1 \in \mathfrak{p}$. Since I is connected in codimension one by Remark 3.2, there exists a chain $\mathfrak{p} = \mathfrak{p}_1, \dots, \mathfrak{p}_r = \mathfrak{q}$ of associated prime ideals of I such that $|\text{supp}(\mathfrak{p}_s) \cap \text{supp}(\mathfrak{p}_{s+1})| = \text{ht}(I) - 1$ for all $1 \leq s \leq r - 1$. As $x_1 \in \mathfrak{p}_1$, by Lemma 3.5 we have $|\text{supp}(\mathfrak{p}_1) \cap \{x_1, \dots, x_t\}| \geq 2$. On the other hand, $|\text{supp}(\mathfrak{p}_1) \cap \text{supp}(\mathfrak{p}_2)| = \text{ht}(I) - 1$. Therefore $|\text{supp}(\mathfrak{p}_2) \cap \{x_1, \dots, x_t\}| \geq 2$. Continuing in this way, we get $|\text{supp}(\mathfrak{q}) \cap \{x_1, \dots, x_t\}| \geq 2$, a contradiction. \square

By Remark 3.2, matroidal ideals connected in codimension one are unmixed. The following example shows that this is not true for polymatroidal ideals which are connected in codimension one.

Example 3.7. The ideal $I = (x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2x_3, x_1x_2^2) = (x_1) \cap (x_1, x_2)^2 \cap (x_1, x_2, x_3)^3$ is polymatroidal which is clearly connected in codimension one, but it is not unmixed.

In our main result Theorem 3.9, we show that an unmixed polymatroidal ideal is connected in codimension one if and only if it is Cohen-Macaulay. We will use the following easy lemma, in our proof.

Lemma 3.8. *Let $I \subset k[x_1, \dots, x_n]$ be an unmixed fully supported polymatroidal ideal with $\text{ht}(I) > 1$. If I is not squarefree, then $\text{ht}(I) \neq n - 1$.*

Proof. Assume that $\text{ht}(I) = n - 1$. Then S/I is not Cohen-Macaulay by Theorem 2.4 and $\dim(S/I) = 1$. Therefore $\text{depth}(S/I) = 0$ and so $\mathfrak{m} \in \text{Ass}(S/I)$ which contradicts $\text{ht}(I) = n - 1$ and the assumption that I is unmixed. \square

Now, we present the main result of this section, which states that

Theorem 3.9. *Let I be an unmixed polymatroidal ideal. Then I is connected in codimension one if and only if I is Cohen-Macaulay.*

Proof. If I is Cohen-Macaulay, then I is connected in codimension one by Remark 3.4. Now let $I = \mathfrak{p}_1^{a_1} \cap \dots \cap \mathfrak{p}_r^{a_r}$ be connected in codimension one. We may assume that I is fully supported with $\text{ht}(I) > 1$ and is not squarefree, by Theorem 2.4 and Theorem 3.6. Therefore to prove that I is Cohen-Macaulay, according to Theorem 2.4, we must show that $r = 1$. We use induction on d , which is the common degree of monomial generators of I . For $d = 2$, the result follows by Lemma 2.2. Let $d > 2$ and $a_i > 1$ for some $1 \leq i \leq r$ and assume contrary that $r > 1$. Since I is connected in codimension one by Remark 3.2, there exist $1 \leq j \neq i \leq r$ such that $|\text{G}(\mathfrak{p}_i) \cap \text{G}(\mathfrak{p}_j)| = \text{ht}(I) - 1$. Note that $\text{ht}(I) \neq n - 1$ by Lemma 3.8 and so $\mathfrak{p} := \mathfrak{p}_i + \mathfrak{p}_j \neq \mathfrak{m}$. On the other hand $\text{supp}(I(\mathfrak{p})) = \text{G}(\mathfrak{p}_i) \cup \{x\}$ for some variable $x \in \mathfrak{p}_j \setminus \mathfrak{p}_i$. Now let $\mathfrak{q} \in \text{Ass}(S/I(\mathfrak{p}))$ and $\mathfrak{q} \neq \mathfrak{p}$. Then $\text{G}(\mathfrak{q}) \subseteq \text{G}(\mathfrak{p}_i) \cup \{x\}$. Since $\text{ht}(\mathfrak{p}_i) = \text{ht}(\mathfrak{q})$, then $\mathfrak{q} = (\text{G}(\mathfrak{p}_i) \setminus \{y_{\mathfrak{q}}\}, x)$ for some variable $y_{\mathfrak{q}}$. Hence $I(\mathfrak{p})$ is a polymatroidal ideal connected in codimension one which is generated in degree less than d . Now, induction assumption implies $|\text{Ass}(S/I(\mathfrak{p}))| = 1$ which is a contradiction. \square

Corollary 3.10. *Let $I \subset S$ be an unmixed fully supported polymatroidal ideal and connected in codimension one. Then $\text{supp}(I(\mathfrak{p}_{\{i\}}))$ is either an empty set or is equal to $\{x_1, \dots, x_n\} \setminus \{x_i\}$ for each $i = 1, \dots, n$.*

Proof. If I is a squarefree Veronese ideal in variables x_1, \dots, x_n , then for all $1 \leq i, j \leq n$, $x_i x_j | u$ for some $u \in \text{G}(I)$. Hence the result is clear by Theorem 3.9 and Theorem 2.4. \square

Corollary 3.11. *Let I be a polymatroidal ideal. Then I satisfies Serre's condition (S_n) for some $n \geq 2$ if and only if I is Cohen-Macaulay.*

Proof. Assume that I satisfies Serre's condition (S_n) for some $n \geq 2$. Then I is connected in codimension one by [5, Corollary 2.4] and Remark 3.3. Hence I is equidimensional by Remark 3.2 and so I is unmixed since it is (S_1) . Now the result follows by Theorem 3.9. \square

4. GENERALIZED COHEN-MACAULAY POLYMATROIDAL IDEALS

A finitely generated module M over a local ring (R, \mathfrak{n}) is called *generalized Cohen-Macaulay*, whenever each local cohomology module $H_{\mathfrak{n}}^i(M)$ has finite length for all $i < \dim M$. It is known that if M is generalized Cohen-Macaulay, then $M_{\mathfrak{p}}$ is Cohen-Macaulay for all prime ideals $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{n}\}$ and the converse holds if R is universally catenary and all its formal fibres are Cohen-Macaulay [1, Exercises 9.5.7 and 9.6.8]. In the following we consider graded generalized Cohen-Macaulay modules over the *local graded polynomial ring (S, \mathfrak{m}) . We call an ideal I generalized Cohen-Macaulay whenever the i th cohomology module $H_{\mathfrak{m}}^i(S/I)$ is of finite length for all $i < \dim(S/I)$.

Lemma 4.1. *The following statements are equivalent for a monomial ideal I .*

- a) I is generalized Cohen-Macaulay.
- b) I is equidimensional and $I(\mathfrak{p})$ is Cohen-Macaulay for all monomial prime ideals $\mathfrak{p} \neq \mathfrak{m}$.

Proof. Note that homogeneous prime ideals of S in multigraded structure are precisely monomial prime ideals. Thus all minimal elements of the non Cohen-Macaulay locus of S/I are monomial by [12, Corollary 3.7]. Now the result follows by [1, Exercise 9.5.7]. \square

Proposition 4.2. *Let I be a polymatroidal ideal generated in degree 2. Then the following statements are equivalent:*

- a) I is equidimensional.
- b) I is generalized Cohen-Macaulay.

Proof. Note that $I(\mathfrak{p}_{\{i\}})$ is generated by indeterminates or is equal to S , and so it is Cohen-Macaulay for all $i = 1, \dots, n$. \square

The following example shows that the above result is not true for polymatroidal ideals generated in degree $d > 2$.

Example 4.3. *The ideal $I = (uxy, uyz, uzv, uxw, xyz, wxz) = (x, u) \cap (x, z) \cap (y, w) \cap (z, u)$ is an unmixed matroidal ideal. But it is not generalized Cohen-Macaulay, since $I(\mathfrak{p}_{\{x\}}) = (uy, uw, yz, wz)$ is not Cohen-Macaulay by Theorem 2.4.*

In the above example, I is an unmixed ideal which is not generalized Cohen-Macaulay. As a main result of this section we will show in Theorem 4.5 that the generalized Cohen-Macaulay matroidal ideals generated in degree $d > 2$ are precisely Cohen-Macaulay matroidal ideals. In order to prove this, we need the following result which is interesting in its own.

Proposition 4.4. *Let $I \subset k[x_1, \dots, x_n]$ be a fully supported matroidal ideal generated in degree $d > 2$. If I is generalized Cohen-Macaulay, then $\operatorname{supp}(I(\mathfrak{p}_{\{i\}})) = \{x_1, \dots, x_n\} \setminus \{x_i\}$ for each $i = 1, \dots, n$.*

Proof. We show, for convenience, that $\operatorname{supp}(I(\mathfrak{p}_{\{1\}})) = \{x_2, \dots, x_n\}$. Let $I(\mathfrak{p}_{\{1\}})$ be fully supported in $K[x_{t+1}, \dots, x_n]$ for some $t \geq 1$. It is enough to show that $t = 1$. Since $I(\mathfrak{p}_{\{1\}})$ is a squarefree Veronese ideal in variables x_{t+1}, \dots, x_n of degree $d - 1$, where d is the common degree of generators of I , it follows that

$$(1) \quad h = \operatorname{ht}(I) = (n - t) - (d - 1) + 1 = n - t - d + 2.$$

Since $d > 2$, we have that for each $j = t + 2, \dots, n$, there exists $u_j \in G(I)$ such that $x_{t+1}x_j|u_j$ and so

$$(2) \quad \{x_{t+2}, \dots, x_n\} \subseteq \text{supp}(I(\mathfrak{p}_{\{t+1\}})).$$

On the other hand since $\text{supp}(I(\mathfrak{p}_{\{1\}})) = \{x_{t+1}, \dots, x_n\}$, it follows that $x_1x_j \nmid u$ for each $j = 2, \dots, t$ and any $u \in G(I)$. So Lemma 3.5 implies that $I(\mathfrak{p}_{\{1\}}) = I(\mathfrak{p}_{\{j\}})$ for $j = 2, \dots, t$. Now again since $\text{supp}(I(\mathfrak{p}_{\{j\}})) = \{x_{t+1}, \dots, n\}$ for $j = 1, \dots, t$, we have that for each $j = 1, \dots, t$, there exists $u_j \in G(I)$ such that $x_jx_{t+1}|u_j$. Thus

$$(3) \quad \{x_1, \dots, x_t\} \subseteq \text{supp}(I(\mathfrak{p}_{\{t+1\}})).$$

Hence by (2) and (3), we have that $\text{supp}(I(\mathfrak{p}_{\{t+1\}})) = \{x_1, \dots, x_n\} \setminus \{x_{t+1}\}$. Therefore since $I(\mathfrak{p}_{\{t+1\}})$ is a squarefree Veronese ideal, it follows that $h = (n - 1) - (d - 1) + 1 = n - d + 1$. Hence from (1), $n - t - d + 2 = n - d + 1$. So $t = 1$. \square

Theorem 4.5. *Let I be a matroidal ideal generated in degree $d > 2$. Then I is generalized Cohen-Macaulay if and only if I is Cohen-Macaulay.*

Proof. By Proposition 4.4, $I(\mathfrak{p}_{\{i\}})$ is a squarefree Veronese ideal in the variables $\{x_1, \dots, x_n\} \setminus \{x_i\}$, for all $1 \leq i \leq n$. Now, since $I = \sum_{i=1}^n x_i I(\mathfrak{p}_{\{i\}})$, the result is clear. \square

The following lemma will be used in the classification of generalized Cohen-Macaulay polymatroidal ideals in Theorem 4.8.

Lemma 4.6. *Let $I = J \cap \mathfrak{m}^d$ be a polymatroidal ideal generated in degree d where J is a squarefree monomial ideal. If $\deg(u) > 1$ for all $u \in G(J)$, then J is a matroidal ideal.*

Proof. Let $u, v \in G(J)$ such that $x_i|u$ and $x_i \nmid v$. Then $x_l|v$ and $x_l \nmid u$ for some $l \neq i$. By assumption there exists $h \neq i$ such that $x_h|u$. Now $u' = x_h^{d-s}u$ and $v' = x_l^{d-r}v$ belong to $G(I)$, where $r = \deg(v)$ and $s = \deg(u)$. Since $\deg_{x_i}(u') > \deg_{x_i}(v')$ and I is polymatroidal ideal, there exists $1 \leq j \neq i \leq n$ such that $\deg_{x_j}(u') < \deg_{x_j}(v')$ and $x_ju'/x_i \in G(I)$. Hence $x_ju'/x_i \in J$. Note that J is squarefree, $x_h|u$ and $h \neq i$, thus $x_ju/x_i \in J$ and also $\deg_{x_j}(u) < \deg_{x_j}(v)$. \square

Lemma 4.7. *Let $I = J \cap \mathfrak{m}^d$ be a monomial ideal generated in degree d where J is a monomial ideal generated in degree $t \leq d$. Then $I = J\mathfrak{m}^{d-t}$.*

Proof. It is clear that $J\mathfrak{m}^{d-t} \subseteq I$. Now let $u \in G(I)$, so there exists $v \in G(J)$ such that $v|u$. So since $\deg(v) = t \leq d = \deg(u)$, there exists a monomial w of degree $d - t$ such that $u = vw$. Hence $u \in J\mathfrak{m}^{d-t}$. \square

Theorem 4.8. *Let $I = J \cap \mathfrak{m}^s$ be a fully supported monomial ideal in $S = K[x_1, \dots, x_n]$ and generated in degree d , where $s \in \{0, d\}$. Then I is a generalized Cohen-Macaulay polymatroidal ideal if and only if one of the following statements holds true:*

- a) J is a Cohen-Macaulay polymatroidal ideal i.e. J is either a principal ideal, a Veronese ideal, or a squarefree Veronese ideal.
- b) $J = \mathfrak{p}_1^{a_1} \cap \dots \cap \mathfrak{p}_r^{a_r}$ is equidimensional and $\mathfrak{p}_i + \mathfrak{p}_j = \mathfrak{m}$ for all $i \neq j$.
- c) J is an unmixed matroidal ideal of degree 2.

Proof. By Lemma 4.7, each of statements (a) or (c) implies that I is polymatroidal. Since I is generated in a single degree, from statement (b) follows that I is polymatroidal by [4, Theorem 3.1].

Whenever (a) holds, I is equidimensional, since J is unmixed. On the other hand for all monomial prime $\mathfrak{p} \neq \mathfrak{m}$, $I(\mathfrak{p}) = J(\mathfrak{p})$ is Cohen-Macaulay.

Assume that (b) holds and let $\mathfrak{q} \in V(I) \setminus \{\mathfrak{m}\}$ be a monomial prime ideal. Since $\mathfrak{p}_i + \mathfrak{p}_j = \mathfrak{m}$ for all $i \neq j$ and $\mathfrak{q} \neq \mathfrak{m}$, we get $I(\mathfrak{q}) = \mathfrak{p}_k^{a_k}$ for some k , $1 \leq k \leq r$ or $I(\mathfrak{q}) = S$.

Assume that (c) holds. By Proposition 4.2, J is generalized Cohen-Macaulay. Thus for all monomial prime $\mathfrak{p} \neq \mathfrak{m}$, $I(\mathfrak{p}) = J(\mathfrak{p})$ is Cohen-Macaulay.

Conversely, assume that I is a generalized Cohen-Macaulay polymatroidal ideal, (a) and (b) don't hold. Note that $J := \mathfrak{p}_1^{a_1} \cap \dots \cap \mathfrak{p}_r^{a_r}$ is an unmixed ideal. Since (b) doesn't hold, let for convenience $\mathfrak{q} = \mathfrak{p}_1 + \mathfrak{p}_2 \neq \mathfrak{m}$. Then $I(\mathfrak{q}) = \mathfrak{p}_1^{a_1} \cap \mathfrak{p}_2^{a_2} \cap \dots \cap \mathfrak{p}_t^{a_t}$ is Cohen-Macaulay for some $2 \leq t \leq r$. Since the unmixed ideal J is not principal, $J(\mathfrak{q}) = I(\mathfrak{q})$ is not principal. Now, Theorem 2.4 implies that $I(\mathfrak{q})$ is squarefree Veronese ideal. Therefore $a_1 = \dots = a_t = 1$ and so $I = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_t \cap \mathfrak{p}_{t+1}^{a_{t+1}} \cap \dots \cap \mathfrak{p}_r^{a_r} \cap \mathfrak{m}^s$.

We claim that $a_{t+1} = \dots = a_r = 1$. Otherwise, there exists $t+1 \leq i \leq r$ such that $a_i \neq 1$. Since $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_i$, there exists a variable $x_l \in \mathfrak{p}_1 \setminus \mathfrak{p}_i$. Note that $x_l \notin \bigcap_{j=1}^t \text{supp}(\mathfrak{p}_j)$, since $I(\mathfrak{q})$ is generated in a single degree and is not a prime ideal. Let $x_l \notin \mathfrak{p}_j$ for $1 \leq j \leq t$. Then $I(\mathfrak{p}_{\{l\}}) = \mathfrak{p}_j \cap \mathfrak{p}_i^{a_i} \cap \mathfrak{q}'$ which is not Cohen-Macaulay. This contradiction implies our claim that $I = J \cap \mathfrak{m}^s$ where J is a squarefree monomial ideal. Since $I(\mathfrak{q})$ is squarefree Veronese ideal of height greater than one, J does not contain any variables since $J(\mathfrak{q}) = I(\mathfrak{q})$. Now, the result follows by Lemma 4.6 and Theorem 4.5, since J is not Cohen-Macaulay. \square

The following examples show that in the above characterization, none of the items (a), (b) or (c) can be removed.

Example 4.9. (i) The ideal $I = (x_1x_2^3, x_1^2x_2^2) = (x_1) \cap (x_2^2) \cap (x_1, x_2)^4$ is polymatroidal which satisfies (a) and (b), but (c) doesn't hold for it.

(ii) The ideal $I = (x_1^2x_2, x_1x_2^2, x_1x_2x_3) = (x_1) \cap (x_2) \cap (x_1, x_2, x_3)^3$ is polymatroidal which satisfies (a) and (c), but (b) doesn't hold for it.

(iii) The ideal $I = (x_1, x_2, x_3, x_4) \cap (x_3, x_4, x_5, x_6) \cap (x_1, x_2, x_5, x_6)$ constructed in [6], is matroidal ideal which satisfies (b) and (c), but (a) doesn't hold for it.

Example 4.10. The ideal $I = (x_1, x_2) \cap (x_2, x_3)^2 \cap (x_1, x_2, x_3)^3$ is polymatroidal by [4, Theorem 3.1] and generalized Cohen-Macaulay by Theorem 4.8 satisfying condition (b), but $J = (x_1, x_2) \cap (x_2, x_3)^2$ is not even generated in a single degree.

Note that the above example is connected in codimension one. There exist polymatroidal ideals connected in codimension one, which are not generalized Cohen-Macaulay, see Example 3.7. In this example the localization $I(\mathfrak{p}_{\{3\}}) = (x_1) \cap (x_1, x_2)^2$ is not Cohen-Macaulay.

Polymatroidal ideals which satisfy condition (c) of Theorem 4.8, can be specified by the following lemma.

Lemma 4.11. *Let I be a fully supported monomial ideal of degree 2. Then I is polymatroidal if and only if $\mathfrak{p}_i + \mathfrak{p}_j = \mathfrak{m}$ for $i \neq j$ and all $\mathfrak{p}_i \in \text{Ass}(S/I)$.*

Proof. Let $\mathfrak{p}_i + \mathfrak{p}_j = \mathfrak{m}$ for $i \neq j$ and all $\mathfrak{p}_i \in \text{Ass}(S/I)$. Since I is generated in a single degree, it follows by [4, Theorem 3.1] that I is polymatroidal. Conversely, Let $I = \mathfrak{p}_1^{a_1} \cap \mathfrak{p}_2^{a_2} \cap \cdots \cap \mathfrak{p}_t^{a_t}$ be polymatroidal and $\mathfrak{q} = \mathfrak{p}_i + \mathfrak{p}_j \neq \mathfrak{m}$ for some $i \neq j$. Then $I(\mathfrak{q}) = \mathfrak{p}_i^{a_i} \cap \mathfrak{p}_j^{a_j} \cap \mathfrak{q}'$ for some monomial ideal \mathfrak{q}' . Since I is generated in degree 2 it follows that the ideal $I(\mathfrak{q})$ is a monomial prime ideal or is equal to S , which is a contradiction. \square

By the above lemma, in the case (c) of Theorem 4.8, for any pair of distinct prime ideals $\mathfrak{p}, \mathfrak{q} \in \text{Ass}(S/J)$ we have $G(\mathfrak{p} + \mathfrak{q}) = \text{supp}(J)$ and $\text{supp}(J)$ is not necessarily equal to the set of all variables. But in the case (b), the same condition holds with the distinctive point that J is fully supported in $\text{supp}(I)$, see Example 4.9(ii).

ACKNOWLEDGMENTS

The authors would like to thank Hossein Sabzrou for fruitful discussions and useful comments regarding this paper.

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